

LOCAL REGULARITY OF THE GREEN OPERATOR IN A CR MANIFOLD OF GENERAL “TYPE”

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ABSTRACT. It is here proved that if a pseudoconvex CR manifold M of hypersurface type has a certain “type”, that we quantify by a vanishing rate F at a submanifold of CR dimension 0, then \square_b “gains f^2 derivatives” where f is defined by inversion of F . Next a general tangential estimate, “twisted” by a pseudodifferential operator Ψ is established. The combination of the two yields a general “ f -estimate” twisted by Ψ , that is, (1.4) below. We apply the twisted estimate for Ψ which is the composition of a cut-off η with a differentiation of order s such as R^s of Section 3. Under the assumption that $[\partial_b, \eta]$ and $[\partial_b, [\bar{\partial}_b, \eta]]$ are superlogarithmic multipliers in a sense inspired to Kohn, we get the local regularity of the Green operator $G = \square_b^{-1}$. In particular, if M has “infraexponential type” along $S \setminus \Gamma$ where S is a manifold of CR dimension 0 and Γ a curve transversal to $T^c M$, then we have local regularity of G . This gives an immediate proof of [1] in tangential version and of [14]. The conclusion extends to “block decomposed” domains for whose blocks the above hypotheses hold separately.

MSC: 32F10, 32F20, 32N15, 32T25

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1. INTRODUCTION

It has been proved in [9] that if the boundary of a pseudoconvex domain of \mathbb{C}^n has geometric “type F ”, then there is an “ f -estimate” for the $\bar{\partial}$ -Neumann problem for $f = F^*(t^{-1})^{-1}$ where F^* is the inverse function to F . The converse is also true (cf. [10]), apart from a loss of accuracy in the estimate which is in most cases negligible. The successful approach in establishing the equivalence between the F -type and the f -estimate consists in triangulating through a potential theoretical condition, namely, the “ f -property”, that is, the existence of a bounded weight whose Levi-form grows with the rate of f^2 at the

boundary. This generalizes former work by Kohn [12], Catlin [5], [6], McNeal [17] et alii. What we prove here is that the F type implies the f -estimate for the tangential system $\bar{\partial}_b$; this is a generalization of Kohn [15]. In greater detail, let $M \subset \mathbb{C}^n$ be a pseudoconvex manifold of hypersurface type and v or u a form in M of a certain degree h . We use the microlocal decomposition into wavelets $u = \sum_{k=1}^{+\infty} \Gamma_k u$ (cf. [15] proof of Theorem 6.1). We consider a submanifold $S \subset M$ of CR dimension 0, and a real function F satisfying $\frac{F}{d_S^2} \searrow 0$ as the distance d_S to S decreases to 0. We also use the notation Id for the identity of the complex tangent bundle $T^{\mathbb{C}}M = TM \cap iTM$. We assume that M has type F along S in a neighborhood U of point $z_o \in S$ in the sense that the Levi form (c_{ij}) of M satisfies $(c_{ij}) \gtrsim \frac{F(d_S)}{d_S^2} \text{Id}$. Then, there is a bounded family of weights $\{\phi^k\}$ by the aid of which we get the estimate of the f -norm by the Levi form (c_{ij}) of M and (ϕ_{ij}^k) of the ϕ^k 's.

Theorem 1.1. *Let M have type F along S ; then*

$$\left\{ \begin{array}{l} \|f(\Lambda)v\| \lesssim \int_M (c_{ij})(\Lambda^{\frac{1}{2}}v, \overline{\Lambda^{\frac{1}{2}}v}) dV + \sum_{k=1}^{+\infty} \int_M (\phi_{ij}^k)(\Gamma_k v, \overline{\Gamma_k v}) dV \\ \quad + \|v\|_0^2, \quad \text{for any } v \text{ of degree } h \in [1, \dim_{CR}(M)], \\ \|f(\Lambda)v\| \lesssim \int_M \left(\text{Trace}(c_{ij})\text{Id} - (c_{ij}) \right) (\Lambda^{\frac{1}{2}}v, \overline{\Lambda^{\frac{1}{2}}v}) dV + \sum_{k=1}^{+\infty} \int_M \left(\text{Trace}(\phi_{ij}^k)\text{Id} - (\phi_{ij}^k) \right) \times \\ \quad \times (\Gamma_k v, \overline{\Gamma_k v}) dV + \|v\|_0^2, \quad \text{for any } v \text{ of degree } h \in [0, \dim_{CR}(M) - 1]. \end{array} \right. \quad (1.1)$$

The proof is the content of Section 2 below. We denote by $u = u^+ + u^- + u^0$ the microlocal decomposition of u (cf. [15] Section 2) and also use the notation Q^b for the energy $Q^b = \|\bar{\partial}_b v\|^2 + \|\bar{\partial}_b^* v\|^2$, and \mathcal{H} for the space of harmonic forms $\mathcal{H} = \ker \bar{\partial}_b \cap \ker \bar{\partial}_b^*$. We apply the first of (1.1) for $v = u^+$, resp. the second for $v = u^-$, and plug into a basic estimate. We also use the elliptic estimate for u^0 and conclude

Theorem 1.2. *We have*

$$\|f(\Lambda)u\|^2 \lesssim Q^b(u, \bar{u}) + \|u\|_0^2, \quad \text{for any } u \text{ of degree } h \in [0, \dim_{CR}(M)]. \quad (1.2)$$

As it has been already said, (1.2) follows from (1.1) for the common range of degrees $h \in [1, \dim_{CR}(M) - 1]$. As for the critical top and bottom degrees, we get the estimate for $u \in \mathcal{H}^\perp$ from the estimate in nearby degree from closed range of $\bar{\partial}_b$ and $\bar{\partial}_b^*$ ([15] proof of Theorem 7.3 p. 237).

Next, we prove a general basic weighted estimate twisted by a pseudodifferential operator Ψ , that is, (3.2) and (3.3) of Theorem 3.1 below. We have to mention that our formula is classical (cf. McNeal [18], [19]) when Ψ is a function. A recent application, in which Ψ is a family of cut-off, has been given in [2] in the problem of the local regularity of the Green operator $G = \square_b^{-1}$. We choose a smooth orthonormal basis of $(1, 0)$ forms

$\omega_1, \dots, \omega_{n-1}$, supplement by a purely imaginary form γ and denote the dual basis of vector fields by $\partial_{\omega_1}, \dots, \partial_{\omega_{n-1}}, T$. We define various constants c_{ij}^h 's as the coefficients of the commutator $[\partial_{\omega_i}, \bar{\partial}_{\omega_j}] = c_{ij}^n T + \sum_{j=1}^{n-1} c_{ij}^h \partial_{\omega_h} - \sum_{j=1}^{n-1} \bar{c}_{ji}^h \bar{\partial}_{\omega_h}$; sometimes, we also write c_{ij} instead of c_{ij}^n . We use the notation $\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}$ for an operator of order smaller than Ψ whose support is contained in a conical neighborhood of that of Ψ . Combination of the f estimate with the basic twisted estimate yields

Theorem 1.3. *Let M have type F along a CR manifold S of CR dimension 0 at z_0 and $U = U_t$ be suitably small. For any form $v = u^+ \in C_c^\infty(M \cap U)$ of degree $h \in [1, \dim_{CR}(M) - 1]$ we have*

$$\begin{aligned} \|f(\Lambda)\Psi v\|^2 &\leq \int (c_{ij})(\Psi T^{\frac{1}{2}}v, \overline{\Psi T^{\frac{1}{2}}v}) dV + \sum_k \int (\phi^k)_{ij}(\Gamma_k \Psi v, \overline{\Gamma_k \Psi v}) dV + t \|\Psi v\|_0^2 \\ &\lesssim Q_\Psi^b(v, \bar{v}) + \|[\partial_b, \Psi] \lrcorner v\|_0^2 + \left| \sum_h \int (c_{ij}^h)([\partial_{\omega_h}, \Psi](v), \overline{\Psi v}) dV \right| \\ &\quad + \left| \int [\partial_b, [\bar{\partial}_b, \Psi^2]](v, \bar{v}) dV \right| + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^b(v, \bar{v}) + \|\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}v\|_0^2 + \|\Psi v\|_0^2. \end{aligned} \quad (1.3)$$

Here $Q_\Psi^b = \|\Psi \bar{\partial}_b v\|^2 + \|\Psi \bar{\partial}_b^* v\|^2$.

(ii) *The similar equation holds for u^- in degree $[0, \dim_{CR}(M) - 1]$ if we replace (c_{ij}) , (ϕ_{ij}^k) and $[\partial_b, [\bar{\partial}_b, \Psi^2]]$ by $-(c_{ij}) + \sum_j c_{jj} Id$, $-(\phi_{ij}^k) + \sum_j \phi_{jj} Id$ and $-[\partial_b, [\bar{\partial}_b, \Psi^2]] + \text{Trace}([\partial_b, [\bar{\partial}_b, \Psi^2]]) Id$ respectively.*

(iii) *Taking summation of the estimate for $v = u^+, v = u^-$ together with the elliptic estimate for $v = u^0$, and using the closed range of $\bar{\partial}_b$ and $\bar{\partial}_b^*$ for the critical degrees we get for the full $u \in \mathcal{H}^\perp$ in degree $h \in [0, \dim_{CR}(M)]$*

$$\begin{aligned} \|f(\Lambda)\Psi u\|_0^2 &\lesssim Q_\Psi^b(u, \bar{u}) + \|[\partial_b, \Psi] \lrcorner u\|_0^2 + \left| \int_M [\partial_b, [\bar{\partial}_b, \Psi^2]](u^+, \bar{u}^+) dV \right| \\ &\quad + \left| \sum_h \int (c_{ij}^h)([\partial_{\omega_h}, \Psi](u), \overline{\Psi u}) dV \right| + \left| \int_M \left(-[\partial_b, [\bar{\partial}_b, \Psi^2]](u^-, \bar{u}^-) \right. \right. \\ &\quad \left. \left. + \text{Trace}([\partial_b, [\bar{\partial}_b, \Psi^2]]) Id \right)(u^-, \bar{u}^-) dV \right| + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^b(u, \bar{u}) + \|\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}u\|_0^2 + \|\Psi u\|_0^2. \end{aligned} \quad (1.4)$$

The proof is just the superposition of the items (i) and (ii) of Theorem 3.1 below. We have indeed, in Theorem 3.1 (i) and (ii) a more general, weighted version of this estimate. We give an application of the general twisted estimate in which Ψ includes a cut-off η and a differentiation of arbitrarily high order s (such as R^s of Section 4 below). To introduce it, we need the notion of superlogarithmic multipliers which are an obvious variant of the subelliptic multipliers (cf. [15] Definition 8.1). The crucial point in our discussion is

that we consider vector multipliers $g = (g_j)$ and also require a more intense property in which energy is replaced by Levi form, that is, for any ϵ , suitable c_ϵ , and for an uniformly bounded family of weights $\{\phi^k\}$

$$\|\log(\Lambda)g \lrcorner v\|^2 \lesssim \epsilon \left(\int_M (c_{ij}(\Lambda^{\frac{1}{2}}v, \overline{\Lambda^{\frac{1}{2}}v}) dV + \sum_{k=1}^{+\infty} \int_M (\phi_{ij}^k)(\Gamma_k v, \overline{\Gamma_k v}) dV \right) + c_\epsilon \|v\|_0^2. \quad (1.5)$$

We also require that the same estimate holds for (c_{ij}) and (ϕ_{ij}^k) replaced by $-(c_{ij}) + \text{Trace}(c_{ij})\text{Id}$ and $-(\phi_{ij}^k) + \text{Trace}(\phi_{ij}^k)\text{Id}$ respectively. With this preliminary we have

Theorem 1.4. *Assume that there is a system of cut-off $\{\eta\}$ at z_o such that $[\bar{\partial}_b, \eta]$ and $[\partial_b, [\bar{\partial}_b, \eta]]$ are vector and matrix superlogarithmic multipliers respectively, and (c_{ij}^h) are subelliptic multipliers. Then G is regular at z_o .*

The proof is found in Section 4. We combine Theorem 1.4 with 1.1. This gives back the conclusion of [1] (in a tangential version) which was in turn a generalization of [14]. It also provides a larger class of hypersurfaces for which G is regular. Let M be the “block decomposed” hypersurface of \mathbb{C}^n defined by $x_n = \sum_{j=1}^m h^{I^j}(z_{I^j}, y_n)$ where $z = (z_{I^1}, \dots, z_{I^m}, z_n)$ is a decomposition of coordinates.

Theorem 1.5. *Assume that*

$$\left\{ \begin{array}{l} (a) \ h^{I^j} \text{ has infraexponential type along a totally real } S^{I^j} \setminus \Gamma^{I^j} \text{ where } S^{I^j} \text{ is} \\ \quad \text{totally real in } \mathbb{C}^{I^j} \times \mathbb{C}_{z_n} \text{ and } \Gamma^{I^j} \text{ is a curve of } \mathbb{C}^{I^j} \times \mathbb{C}_{z_n} \text{ transversal to } \mathbb{C}^{I^j} \times \{0\}, \\ (b) \ h_{z_j}^j \text{ are superlogarithmic multipliers,} \\ (c) \ c_{ij}^h \text{ are subelliptic multipliers.} \end{array} \right. \quad (1.6)$$

Then, we have local regularity of G at $z_o = 0$.

In case of a single block $x_n = h^{I^1}$ we regain [2] and [14]. The proof is found in Section 4 below.

Example Let

$$(i) \quad x_n = \sum_{j=1}^{n-1} e^{-\frac{1}{|z_j|^a}} e^{-\frac{1}{|x_j|^b}} \quad \text{for any } a \geq 0 \text{ and for } b < 1.$$

Then, (1.6) (a) is obtained starting from $h_{z_j \bar{z}_j}^j > \frac{e^{-\frac{1}{|x_j|^b}}}{|x_j|^2}$, that is, the condition of type $F_j := e^{-\frac{1}{|x_j|^b}}$ along $S_j = \mathbb{R}_{y_j} \times \{0\}$. This yields the estimate of the f norm for $f(t) = \log^{\frac{1}{b}}(t)$; since $\frac{1}{b} > 1$, this is superlogarithmic. (1.6) (b) follows from $|h_{z_j}^j|^2 \lesssim h_{z_j \bar{z}_j}^j$ which says that the $h_{z_j}^j$'s are not only superlogarithmic, but indeed $\frac{1}{2}$ -subelliptic, multipliers.

Finally, (c) follows from $c_{jj}^h \lesssim c_{jj}$ (a consequence of the “rigidity” of M) which shows that these constant are $\frac{1}{2}$ subelliptic multipliers.

(1.6) is the ultimate step of a long sequence of criteria of regularity of G , not reduceable in one another, described by the hypersurface models below, in which $a > 0$ and $0 < b < 1$,

- (ii) $x_n = \sum_{j=1}^{n-1} e^{-\frac{1}{|x_j|^b}}$ Kohn [15],
- (iii) $x_n = e^{-\sum_{j=1}^{n-1} \frac{1}{|z_j|^a}}$ Kohn [14],
- (iv) $x_n = e^{-\frac{1}{\sum_{j=1}^{n-1} |x_j|^a}} \left(\sum_{j=1}^{n-1} e^{-\frac{1}{|x_j|^b}} \right)$ Baracco-Khanh-Zampieri [1],
- (v) $x_n = \sum_{j=1}^{n-1} e^{-\frac{1}{|z_j|^a}}$ Baracco-Pinton-Zampieri [3],
- (vi) $x_n = \sum_{j=1}^{n-1} e^{-\frac{1}{|z_j|^a}} x_j^a$ Baracco-Pinton-Zampieri [2].

Thus, the degeneracy in our model (i) comes as the combination of those of (ii) with (v) (or (vi)).

2. ESTIMATE OF THE f -NORM BY THE LEVI FORM

Let M be a C^∞ CR-manifold of \mathbb{C}^n of hypersurface-type, z_o a point of M , U an open neighborhood of z_o . Our setting being local, we can find a local CR-diffeomorphism which reduces M to a hypersurface of $TM + iTM$; therefore, it is not restrictive to assume that M is a hypersurface of \mathbb{C}^n from the beginning. We choose a smooth orthonormal basis of $(1, 0)$ forms $\omega_1, \dots, \omega_{n-1}$, supplement by a purely imaginary form γ and denote the dual basis of vector fields by $\partial_{\omega_1}, \dots, \partial_{\omega_{n-1}}, T$. We also use the notation $\bar{\partial}_b$ for the tangential CR-system. For a smooth real function ϕ , we denote by (ϕ_{ij}) the matrix of the Levi form $\partial_b \bar{\partial}_b \phi$. Note that ϕ_{ij} differs from $\partial_{\omega_i} \bar{\partial}_{\omega_j}(\phi)$ because of the presence of the derivatives of the coefficients of the forms $\bar{\partial}_{\omega_j}$. Let $(c_{i\bar{j}})_{i,j=1,\dots,n-1}$ be the Levi-form $d\gamma|_{T^{\mathbb{C}}M}$ where $T^{\mathbb{C}}M = TM \cap iTM$.

Let $S \subset M$ be a submanifold of CR-dimension 0, d_S the Euclidean distance to S , and $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a smooth monotonic increasing function such that $f \lesssim t^{\frac{1}{2}}$. We use the notation a_k for the constant $a_k := f^{-1}(2^k)$ and S_{a_k} for the strip $S_{a_k} := \{z \in M : d_S(z) \leq a_k\}$.

Lemma 2.1. *There is an uniformly bounded family of smooth weights $\{\phi^k\}$ with $\text{supp } \phi^k \subset S_{2a_k}$ whose Levi-form satisfies*

$$\partial_b \bar{\partial}_b \phi^k \gtrsim \begin{cases} f^2(2^k) & \text{on } S_{a_k} \\ -f^2(2^k) & \text{on } S_{2a_k} \setminus S_{a_k}, \\ 0 & \text{on } M \setminus S_{2a_k}. \end{cases} \quad (2.1)$$

This also readily implies the same inequalities as (2.1) with $\partial_b \bar{\partial}_b \phi^k$ replaced by $\left(\text{Trace}(\partial_b \bar{\partial}_b \phi^k) \text{Id} - \partial_b \bar{\partial}_b \phi^k \right)$.

Note that there is no assumption about the behavior of M at S in this Lemma.

Proof. Set

$$\phi^k = c\chi\left(\frac{d_S(z)}{a_k}\right) \log\left(\frac{d_S^2(z)}{a_k^2} + 1\right), \quad (2.2)$$

where c is a constant that will be specified later and $\chi \in C^\infty(0, 2)$ is a decreasing cut-off function which satisfies

$$\begin{cases} \chi \equiv 1 & \text{on } [0, 1], \\ 0 \leq \chi \leq 1 & \text{on } [1, \frac{3}{2}], \\ \chi \equiv 0 & \text{on } [\frac{3}{2}, 2]. \end{cases}$$

Remark that

$$\begin{aligned} \partial_b \bar{\partial}_b d_S^2 &= 2\partial_b d_S \otimes \bar{\partial}_b d_S + 2d_S \partial_b \bar{\partial}_b d_S \\ &\geq 2\partial_b d_S \otimes \bar{\partial}_b d_S \\ &\geq \text{Id}, \end{aligned}$$

where the last inequality follows from $\dim_{CR}(M) = 0$ (with the agreement that Id denotes the identity of $T^{\mathbb{C}}M$).

Now, when $\partial_b \bar{\partial}_b$ hits \log , we have

$$\begin{aligned} \partial_b \bar{\partial}_b \log\left(\frac{d_S^2(z)}{a_k^2} + 1\right) &\gtrsim \frac{\partial_b d_S \otimes \bar{\partial}_b d_S + d_S \partial_b \bar{\partial}_b d_S}{a_k^2} \\ &\gtrsim \frac{\text{Id}}{a_k^2} = f^2(2^k) \text{Id}. \end{aligned} \quad (2.3)$$

On the other hand, on S_{a_k} , the function χ is constant and therefore $\partial_b \bar{\partial}_b \phi^k = \partial_b \bar{\partial}_b \log$. Thus (2.3) yields the first of (2.1). When, instead, $\partial_b \bar{\partial}_b$ hits χ , we have

$$\begin{aligned} \left| \partial_b \bar{\partial}_b \chi\left(\frac{d_S(z)}{a_k}\right) \right| &\leq |\ddot{\chi}| \frac{\partial_b d_S \otimes \bar{\partial}_b d_S}{a_k^2} + |\dot{\chi}| \frac{\partial_b \bar{\partial}_b d_S}{a_k} \\ &\lesssim \frac{\text{Id}}{a_k^2}. \end{aligned} \quad (2.4)$$

since $\dim_{CR}(S) = 0$

On the other hand, \log stays bounded on S_{2a_k} and therefore $\partial_b \bar{\partial}_b(\chi) \log \gtrsim -a_k^{-2} = -f^2(2^k)$. Finally, when ∂_b and $\bar{\partial}_b$ hit χ and \log separately, we get

$$\begin{aligned} \left| 2\Re \partial_b \chi \left(\frac{d_S}{a_k} \right) \bar{\partial}_b \log \left(\frac{d_S^2}{a_k^2} + 1 \right) \right| &\lesssim \left| 2\Re \dot{\chi} \frac{\partial_b d_S}{a_k} \otimes \frac{2a_k^2 d_S \bar{\partial}_b d_S}{2d_S^2 a_k^2} \right| \\ &\lesssim \frac{\partial_b d_S \otimes \bar{\partial}_b d_S}{a_k^2} = f^2(2^k) \text{Id}. \end{aligned} \quad (2.5)$$

since $d_S \sim a_k$ on $\text{supp } \dot{\chi}$

Thus, again, $2\Re \bar{\partial}_b \chi \bar{\partial}_b \log \gtrsim -f^2(2^k) \text{Id}$. □

As we have seen in the proof of Lemma 2.1, when $\dot{\chi}$ and $\ddot{\chi} \neq 0$, the Levi form of ϕ^k can get negative. However, this annoyance can be well behaved by the aid of the Levi form of M . Let F be a smooth real function such that $\frac{F(d)}{d^2} \searrow 0$ as $d \searrow 0$, denote by F^* the inverse to F and define $f(t) := (F^*(\delta))^{-1}$, for $\delta = t^{-1}$. Let $f(\Lambda)$ be the tangential pseudodifferential operator with symbol f . This is defined by introducing a local straightening $M \simeq \mathbb{R}^{2n-1} \times \{0\}$ for a defining function $r = 0$ of M , taking local coordinates $x \in M$, dual coordinates ξ of x and setting

$$f(\Lambda)(u) = \int \left(e^{ix\xi} f(\sqrt{1+\xi^2}) \int e^{-iy\xi} u(y) dy \right) d\xi.$$

In particular Λ is the standard elliptic pseudodifferential operator with symbol $\sqrt{1+\xi^2}$.

Definition 2.2. We say that M has type F along S in a neighborhood U of z_o , if

$$(c_{ij}) \gtrsim \frac{F(d_S)}{d_S^2} \text{Id} \quad \text{on } U. \quad (2.6)$$

Note that (2.6) implies

$$\left(\text{Trace}(c_{ij}) \text{Id} - (c_{ij}) \right) \gtrsim \frac{F(d_S)}{d_S^2} \text{Id} \quad \text{on } U. \quad (2.7)$$

Proposition 2.3. *Let M have type F along S of CR dimension 0. Then*

$$\begin{cases} \|f(\Lambda) \Gamma_k v\|_0^2 \lesssim \int_M (c_{ij}) (\Gamma_k \Lambda^{\frac{1}{2}} v, \overline{\Gamma_k \Lambda^{\frac{1}{2}} v}) dV + \int_M (\phi_{ij}^k) (\Gamma_k v, \overline{\Gamma_k v}) dV + \|\Gamma_k v\|_0^2, & h \in [1, n-1], \\ \|f(\Lambda) \Gamma_k v\|_0^2 \lesssim \int_M \left(\text{Trace}(c_{ij}) \text{Id} - (c_{ij}) \right) (\Gamma_k \Lambda^{\frac{1}{2}} v, \overline{\Gamma_k \Lambda^{\frac{1}{2}} v}) dV \\ \quad + \int_M \left(\text{Trace}(\phi_{ij}^k) \text{Id} - (\phi_{ij}^k) \right) (\Gamma_k v, \overline{\Gamma_k v}) dV + \|\Gamma_k v\|_0^2, & h \in [0, n-2]. \end{cases} \quad (2.8)$$

Proof. We set $a_k = f^{-1}(2^k) = F^*(2^{-k})$, $S_{a_k} = \{z : d_S(z) < a_k\}$ and denote by $\lambda(z)$ the minimum of the $n-1$ eigenvalues of (c_{ij}) at z . We start from the first of (2.8). We

have

$$\begin{aligned}
\|\Gamma_k v\|_0^{M \setminus S_{a_k}} &\leq \max_{z \in M \setminus S_{a_k}} \frac{2^{-\frac{k}{2}}}{\lambda(z)^{\frac{1}{2}}} \left(\|\lambda^{\frac{1}{2}} \Gamma_k \Lambda^{\frac{1}{2}} v\|^{M \setminus S_{a_k}} + \|\Gamma_k v\|_{-\frac{1}{2}} \right) \\
&\lesssim \frac{a_k 2^{-\frac{k}{2}}}{F(a_k)^{\frac{1}{2}}} \left(\sqrt{\int_{M \setminus S_{a_k}} (c_{ij})(\Gamma_k \Lambda^{\frac{1}{2}} v, \overline{\Gamma_k \Lambda^{\frac{1}{2}} v}) dV} + 2^{-\frac{k}{2}} \|\Gamma_k v\|_0^{M \setminus S_{a_k}} \right) \\
&\lesssim f^{-1}(2^k) \left(\sqrt{\int_{M \setminus S_{a_k}} (c_{ij})(\Gamma_k \Lambda^{\frac{1}{2}} v, \overline{\Gamma_k \Lambda^{\frac{1}{2}} v}) dV} + \|\Gamma_k v\|_0 \right).
\end{aligned} \tag{2.9}$$

Recalling that $f(\Lambda_\xi) \equiv f(2^k)$ on $\text{supp } \Gamma_k$, this gives

$$\|f(\Lambda) \Gamma_k v\|_0^{M \setminus S_{a_k}} \lesssim \sqrt{\int_{M \setminus S_{a_k}} (c_{ij})(\Gamma_k \Lambda^{\frac{1}{2}} v, \overline{\Gamma_k \Lambda^{\frac{1}{2}} v}) dV} + \|\Gamma_k v\|_0. \tag{2.10}$$

Now, on $S_{2a_k} \setminus S_{a_k}$, $(\phi^k)_{ij}$ can get negative. However, using the second of (2.1) and tuning the choice of c , independent of k , in the definition of ϕ^k so that $2^k(c_{ij}) + (\phi^k)_{ij} \geq \frac{f^2(2^k)}{2}$ on $S_{2a_k} \setminus S_{a_k}$, we have that not only (2.10) but also (2.8) holds on $M \setminus S_{a_k}$.

Finally, on S_{a_k} , $(\phi^k)_{ij}$ satisfies the first of (2.1) and therefore

$$\|f(\Lambda) \Gamma_k v\|_0^{S_{a_k}} \lesssim \sqrt{\int_{S_{a_k}} (\phi_{ij}^k)(\Gamma_k v, \overline{\Gamma_k v}) dV} + \|\Gamma_k v\|_0.$$

This shows how (2.8) follows from (2.6). In the same way we can see that the second follows from (2.7). □

Proof of Theorems 1.1 and 1.2. The proof of (1.1) just consists in taking summation over k in (2.8). As for (1.2) in degrees $h \in [1, n-2]$, it follows from the combination of the first (resp. the second) of (1.1) for $v = u^+$ (resp. $v = u^-$), in addition to the classical basic tangential estimates and the elliptic estimate for u^0 . As for the critical degree $h = 0$ and $h = n-1$ in (1.2), it follows from writing $u = \bar{\partial}_b^* w$ and $u = \bar{\partial}_b w$ respectively (by closed range) and by using the estimate already established for w in the non-critical degrees 1 and $n-2$ respectively.

3. THE TANGENTIAL HÖRMANDER-KOHN-MORREY FORMULA TWISTED BY A PSEUDODIFFERENTIAL OPERATOR

Let M be a CR manifold of hypersurface type of \mathbb{C}^n , $\bar{\partial}_b$ the tangential Cauchy-Riemann system, $\bar{\partial}_b^*$ the adjoint system. Our discussion is local and we can therefore assume that M is in fact a hypersurface. For a neighborhood U of a point $z_o \in M$, we identify $U \cap M$ to \mathbb{R}^{2n-1} with coordinates x and dual coordinates ξ , and consider a pseudodifferential operator Ψ with symbol $\mathcal{S}(\Psi)(x, \xi)$. For notational convenience we assume that the symbol is real. We also use the notation L_ϕ^2 for the L^2 space weighted by $e^{-\phi}$, $Q^b = \|\bar{\partial}_b u\|^2 +$

$\|\bar{\partial}_b^* u\|^2$ for the energy, and $Q_\Psi^{b\phi} = \|\Psi \bar{\partial}_b u\|_\phi^2 + \|\Psi \bar{\partial}_b^* u\|_\phi^2$ for the energy weighted by ϕ and twisted by Ψ . We consider the pseudodifferential decomposition of the identity by Kohn $\text{Id} = \Phi^+ + \Phi^- + \Phi^0$ modulo $\text{Op}^{-\infty}$. We consider a basis of $(1, 0)$ forms $\omega_1, \dots, \omega_{n-1}$ the conjugate basis $\bar{\omega}_1, \dots, \bar{\omega}_{n-1}$ and complete by a purely imaginary form γ . We denote by $\partial_{\omega_1}, \dots, \partial_{\omega_{n-1}}, \bar{\partial}_{\omega_1}, \dots, \bar{\partial}_{\omega_{n-1}}, T$ the dual basis of vector fields. M being a hypersurface defined, say, by $r = 0$, we can supplement the ω_j 's to a full basis of $(1, 0)$ forms in \mathbb{C}^n by adding $\omega_n = \partial r$. Then $\gamma = \omega_n - \bar{\omega}_n$ and $T = \partial_{\omega_n} - \bar{\partial}_{\bar{\omega}_n}$. We describe the commutators by

$$\begin{aligned} [\partial_{\omega_i}, \bar{\partial}_{\bar{\omega}_j}] &= \sum_{j=1}^n c_{ij}^h \partial_{\omega_h} - \sum_{j=1}^n \bar{c}_{ji}^h \bar{\partial}_{\omega_h} \\ &= c_{ij}^n T + \sum_{j=1}^{n-1} c_{ij}^h \partial_{\omega_h} - \sum_{j=1}^{n-1} \bar{c}_{ji}^h \bar{\partial}_{\omega_h}; \end{aligned} \quad (3.1)$$

We also write c_{ij} instead of c_{ij}^n .

For a cut-off $\eta \in C_c^\infty(U \cap M)$ we write $u^+ := \eta \Phi^+ u$, $u^- = \eta \Phi^- u$, $u^0 = \eta \Phi^0 u$, $T^{\pm} = \eta T \Phi^{\pm}$. We note that $\mathcal{S}(T) > 0$ on $\text{supp } \mathcal{S}(\Phi^+)$ (resp. $\mathcal{S}(T^-) > 0$ on $\text{supp } \mathcal{S}(\Phi^-)$) and therefore $T^{\frac{1}{2}}$ (resp. $(T^-)^{\frac{1}{2}}$) makes sense when acting on u^+ (resp. u^-). We make the relevant remark that

$$\begin{cases} \mathcal{S}(T) \sim \Lambda \text{ on } \text{supp } \mathcal{S}(\Phi^+), & \mathcal{S}(T^-) \sim \Lambda \text{ on } \text{supp } \mathcal{S}(\Phi^-), \\ \{\mathcal{S}(\partial_{\omega_j}\}_{j=1, \dots, n-1} \sim \Lambda \text{ and } \mathcal{S}(\bar{\partial}_{\omega_j}\}_{j=1, \dots, n-1} \sim \Lambda & \text{ on } \text{supp } \mathcal{S}(\Phi^0). \end{cases}$$

We denote by $\text{Op}^{\text{ord}(\Psi) - \frac{1}{2}}$, resp. Op^0 , an operator of order $2\text{ord}(\Psi) - \frac{1}{2}$, resp. 0, whose support is contained in $\text{supp } \Psi$; we also assume that Op^0 only depends on the C^2 -norm of M and, in particular, is independent of ϕ and Ψ .

Theorem 3.1. (i) *We have for every smooth form $v = u^+$ of degree $h \in [1, n-1]$*

$$\begin{aligned} & \int_M e^{-\phi} (c_{ij}) (T^{\frac{1}{2}} \Psi v, \overline{T^{\frac{1}{2}} \Psi v}) dV + \int_M e^{-\phi} \left((\phi_{ij}) - \frac{1}{2} (c_{ij}) T(\phi) \right) (\Psi v, \overline{\Psi v}) dV + \|\Psi \bar{\nabla} v\|_\phi^2 \\ & \lesssim Q_\Psi^{b\phi}(v, \bar{v}) + \|[\partial_b, \Psi] \lrcorner v\|_\phi^2 + \|[\partial_b, \phi] \lrcorner \Psi v\|_\phi^2 + \left| \sum_{h=1}^{n-1} \int (c_{ij}^h) ([\partial_{\omega_h}, \Psi](v), \overline{\Psi v}) dV \right| \\ & + \left| \int_M e^{-\phi} [\partial_b, [\bar{\partial}_b, \Psi^2]](v, \bar{v}) dV \right| + Q_{\text{Op}^{\text{ord}(\Psi) - \frac{1}{2}}}^{b\phi}(v, \bar{v}) + \|\text{Op}^{\text{ord}(\Psi) - \frac{1}{2}} v\|_\phi^2 + \|\Psi v\|_\phi^2. \end{aligned} \quad (3.2)$$

Here we are using the notation $Q_\Psi^{b\phi} = \|\Psi \bar{\partial}_b v\|_\phi^2 + \|\Psi \bar{\partial}_b^* v\|_\phi^2$.

(ii) We also have, for $v = u^-$ smooth of degree $h \in [0, n-2]$

$$\begin{aligned}
& \int_M e^{-\phi} \left(- (c_{ij}) + \sum_j c_{jj} Id \right) ((T^-)^{\frac{1}{2}} \Psi v, \overline{(T^-)^{\frac{1}{2}} \Psi v}) + \|\Psi \nabla v\|_\phi^2 \\
& + \int_M e^{-\phi} \left(\left(-(\phi_{ij}) + \sum_j \phi_{jj} Id \right) + \frac{1}{2} \left((c_{ij}) T(\phi) - (\sum_j c_{jj}) T(\phi) \right) \right) (\Psi v, \overline{\Psi v}) dV \\
& \lesssim Q_\Psi^{b\phi}(v, \bar{v}) + \|[\partial_b, \Psi] \lrcorner v\|_\phi^2 + \|[\partial_b, \phi] \lrcorner \Psi v\|_\phi^2 + \left| \sum_{h=1}^{n-1} \int \left(- (c_{ij}^h) + \sum_j c_{jj}^h Id \right) ([\partial_{\omega_h}, \Psi](v), \times \right. \\
& \times \overline{\Psi v}) dV \Big| + \left| \int_M e^{-\phi} \left(- [\partial_b, [\bar{\partial}_b, \Psi^2]] + \text{Trace}([\partial_b, [\bar{\partial}_b, \Psi^2]]) Id \right) (v, \bar{v}) dV \right| \\
& + Q_{Op^{ord(\Psi)-\frac{1}{2}}}^{b\phi}(v, \bar{v}) + \|Op^{ord(\Psi)-\frac{1}{2}} v\|_\phi^2 + \|\Psi v\|_\phi^2.
\end{aligned} \tag{3.3}$$

Clearly u^0 is subject to elliptic estimates. These, combined with (3.2), (3.3) yield an estimate for the full u in degrees $[1, n-2]$ and then also for $u \in \mathcal{H}^\perp$ in degree $k \in [0, n-1]$ by closed range.

Remark 3.2. The formula also holds for Ψ complex: in this case one replaces Ψ^2 by $|\Psi|^2$ and add the additional error term $[\partial_b, \bar{\Psi}] \lrcorner$ to the already existing $[\partial_b, \Psi] \lrcorner$.

Proof. We start from

$$\begin{aligned}
\partial_b \bar{\partial}_b \phi &= \partial_b \left(\sum_j \bar{\partial}_{\omega_j}(\phi) \bar{\omega}_j \right) \\
&= \sum_{ij} \left(\partial_{\omega_i} \bar{\partial}_{\omega_j}(\phi) + \sum_h \bar{c}_{ji}^h \bar{\partial}_{\omega_h}(\phi) \right) \omega_i \wedge \bar{\omega}_j.
\end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}
\bar{\partial}_b \partial_b \phi &= \bar{\partial}_b \left(\sum_j \partial_{\omega_j}(\phi) \omega_j \right) \\
&= \sum_{ij} \left(-\bar{\partial}_{\omega_j} \partial_{\omega_i}(\phi) - \sum_h c_{ij}^h \partial_{\omega_h}(\phi) \right) \omega_i \wedge \bar{\omega}_j.
\end{aligned} \tag{3.5}$$

Differently from the ambient $\bar{\partial}$ -system on \mathbb{C}^n , we do not have $\partial_b \bar{\partial}_b = \bar{\partial}_b \partial_b$ and in fact, combining (3.4) with (3.5), we can describe (ϕ_{ij}^b) , the matrix of $\frac{1}{2}(\partial_b \bar{\partial}_b - \bar{\partial}_b \partial_b)(\phi)$, by

$$\begin{aligned}
\phi_{ij}^b &= \left\langle \frac{1}{2}(\partial_b \bar{\partial}_b - \bar{\partial}_b \partial_b)(\phi), \partial_{\omega_i} \wedge \bar{\partial}_{\omega_j} \right\rangle \\
&\stackrel{\text{by (3.4), (3.5)}}{=} \frac{1}{2} \left(\left(\partial_{\omega_i} \bar{\partial}_{\omega_j} + \bar{\partial}_{\omega_j} \partial_{\omega_i} \right)(\phi) + \sum_{h=1}^{n-1} \bar{c}_{ji}^h \bar{\partial}_{\omega_h}(\phi) + c_{ij}^h \partial_{\omega_h}(\phi) \right) \\
&= \bar{\partial}_{\omega_j} \partial_{\omega_i}(\phi) + \frac{1}{2} \left([\partial_{\omega_i}, \bar{\partial}_{\omega_j}](\phi) + \sum_h \bar{c}_{ji}^h \bar{\partial}_{\omega_h}(\phi) + \sum_h c_{ij}^h \partial_{\omega_h}(\phi) \right) \\
&\stackrel{(3.1)}{=} \bar{\partial}_{\omega_j} \partial_{\omega_i}(\phi) + \frac{1}{2} c_{ij} T(\phi) + \sum_h c_{ij}^h \partial_{\omega_h}(\phi).
\end{aligned} \tag{3.6}$$

We consider now

$$e^\phi \Psi^{-2} [\bar{\partial}_{\omega_i}, e^{-\phi} \Psi^2] = -\phi_{\omega_i} + 2 \frac{[\bar{\partial}_{\omega_i}, \Psi]}{\Psi} + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2}, \tag{3.7}$$

whose sense is fully clear when both sides are multiplied by Ψ^2 . In other terms, we have

$$\bar{\partial}_{e^{-\phi} \Psi^2}^* = \bar{\partial}^* + \partial \phi \lrcorner - 2 \frac{[\partial, \Psi]}{\Psi} \lrcorner + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0. \tag{3.8}$$

This leads us to define the transposed operator δ_{ω_i} to $\bar{\partial}_{\omega_i}$ by

$$\delta_{\omega_i} := \partial_{\omega_i} - \phi_{\omega_i} + 2 \frac{[\partial_{\omega_i}, \Psi]}{\Psi} + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0. \tag{3.9}$$

With these preliminaries we have

$$\begin{aligned}
[\delta_{\omega_i}, \bar{\partial}_{\omega_j}] &= c_{ij} T + \sum_{h=1}^{n-1} c_{ij}^h \delta_{\omega_h} - \sum_{h=1}^{n-1} \bar{c}_{ji}^h \bar{\partial}_{\omega_h} + \left(\phi_{ij}^b - \frac{1}{2} c_{ij} T(\phi) \right) \\
&\quad - 2 \sum_h c_{ij}^h \frac{[\partial_{\omega_h}, \Psi]}{\Psi} + \frac{[\partial_{\omega_i}, [\bar{\partial}_{\omega_j}, \Psi]]}{\Psi} + \frac{[\partial_{\omega_i}, \Psi] \otimes [\bar{\partial}_{\omega_j}, \Psi]}{\Psi^2} + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0.
\end{aligned} \tag{3.10}$$

We remember now that there are two equally reasonable definition of the pseudodifferential action

$$\Psi(w) = \begin{cases} (i) & \int e^{ix\xi} \mathcal{S}(\Psi)(x, \xi) \tilde{w}(\xi) d\xi \\ (ii) & \int e^{ix\xi} (\widetilde{\mathcal{S}(\Psi)}(\cdot, \xi) * \tilde{w}) d\xi, \end{cases} \tag{3.11}$$

where \tilde{w} denotes the Fourier transform. Up to error terms of type $\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}$, we have

$$\begin{aligned}
||\Psi(w)||^2 &\sim (\Psi w, \Psi w) \\
&\stackrel{\text{Plancherel and (3.11) (ii)}}{\sim} (\widetilde{\Psi(w)}, \widetilde{\mathcal{S}(\Psi)}(\cdot, \xi) * \tilde{w}) \\
&\sim \int \widetilde{\Psi(w)}(\xi) \overline{\int \widetilde{\mathcal{S}(\Psi)}(\xi - \eta, \xi) \tilde{w}(\eta) d\eta d\xi} \\
&\stackrel{\text{Plancherel}}{=} \int \widetilde{\Psi(w)}(\xi) \overline{\widetilde{\mathcal{S}(\Psi)}(\eta - \xi, \xi)} \tilde{w}(\eta) d\eta \\
&\stackrel{(3.11) (i)}{\sim} \int \widetilde{\Psi \Psi(w)}(\eta) \tilde{w}(\eta) d\eta \\
&\stackrel{\text{Plancherel}}{\sim} (|\Psi|^2 w, w).
\end{aligned}$$

For the same reason $(\Psi^2 w, w) \sim \int |\Psi|^2 |w|^2 dV$ and therefore

$$||\Psi(w)||^2 \sim \int |\Psi|^2 |w|^2 dV.$$

Adding the weight ϕ and recalling that in our discussion Ψ is real,

$$||\Psi \bar{\partial}_b^{(*)} v||_\phi^2 = \int e^{-\phi} \Psi^2 |\bar{\partial}_b^{(*)} v|^2 dV + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}(\bar{\partial}^{(*)} v)||_\phi^2, \quad (3.12)$$

where $\bar{\partial}_b^{(*)}$ denotes either $\bar{\partial}_b$ or $\bar{\partial}_b^*$. We are ready for the proof of (3.2); we prove it only for $v = u^+$, the proof of (3.3) for $v = u^-$ being similar. We have

$$\begin{aligned}
&\int_\Omega e^{-\phi} (c_{ij})(T\Psi v, \overline{\Psi v}) + \int_\Omega [\partial_b, [\bar{\partial}_b, e^{-\phi} \Psi^2]](v, \bar{v}) dV \\
&\quad - ||[\partial_b, \phi] \lrcorner \Psi v||_\phi^2 - ||[\partial_b, \Psi] \lrcorner v||_\phi^2 + ||\Psi \bar{\nabla} v||_\phi^2 \\
&\leq ||\Psi \bar{\partial}_b v||_\phi^2 + ||\Psi (\bar{\partial}_b)^*_{e^{-\phi} \Psi^2} v||_\phi^2 + sc ||\Psi \bar{\nabla} v||_\phi^2 + \left| \sum_h \int_\Omega e^{-\phi} (c_{ij}^h)([\partial_{\omega_h}, \Psi] v, \overline{\Psi v}) dV \right| \\
&\quad + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^{b\phi}(v, v) + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} v||_\phi^2 + ||\Psi v||_\phi^2, \quad (3.13)
\end{aligned}$$

or, according to (3.10) and after absorbing the term which comes with sc,

$$\begin{aligned}
& \int_M e^{-\phi} c_{ij}(T\Psi v, \overline{\Psi v}) dV + \int_M e^{-\phi} \phi_{ij}(\Psi v, \overline{\Psi v}) dV - ||[\partial_b, \phi] \lrcorner \Psi v||_\phi^2 \\
& + \int_M e^{-\phi} [\partial_i, [\bar{\partial}_j, \Psi^2]](v, \bar{v}) dV - ||[\partial_b, \Psi] \lrcorner v||_\phi^2 + ||\Psi \bar{\nabla} v||_\phi^2 \\
& \lesssim ||\Psi \bar{\partial}_b v||_\phi^2 + ||\Psi (\bar{\partial}_b)^*_{e^{-\phi} \Psi^2} v||_\phi^2 + \left| \sum_h \int_\Omega e^{-\phi} (c_{ij}^h)([\partial_{\omega_h}, \Psi] v, \overline{\Psi v}) dV \right| \\
& + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}(v, v) + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} v||_\phi^2 + ||\Psi v||_\phi^2. \quad (3.14)
\end{aligned}$$

To carry out our proof we need to replace $(\bar{\partial}_b)^*_{e^{-\phi} \Psi^2}$ by $\bar{\partial}_b^*$. We have from (3.10)

$$\begin{aligned}
||\Psi (\bar{\partial}_b)^*_{e^{-\phi} \Psi^2} v||_\phi^2 & \leq ||\Psi \bar{\partial}_b^* v||_\phi^2 + ||\Psi \partial \phi \lrcorner \Psi^2 v||_\phi^2 + ||[\partial_b, \Psi] \lrcorner v||_\phi^2 + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} v||_\phi^2 \\
& + 2 \underbrace{\left| \Re e(\Psi \bar{\partial}_b^* v, \overline{\Psi \partial_b \phi \lrcorner v})_\phi \right| + 2 \left| \Re e(\Psi \bar{\partial}_b^* v, [\partial_b, \Psi] \lrcorner v)_\phi \right| + 2 \left| \Re e(\Psi \partial_b \phi \lrcorner v, [\partial_b, \Psi] \lrcorner v)_\phi \right|}_{\#}. \quad (3.15)
\end{aligned}$$

We next estimate by Cauchy-Schwarz inequality

$$\# \lesssim ||\Psi \bar{\partial}_b^* v||_\phi^2 + ||\Psi \partial_b \phi \lrcorner v||_\phi^2 + ||[\partial_b, \Psi] \lrcorner v||_\phi^2.$$

We move the third, forth and fifth terms from the left to the right of (3.14), and get (3.2) with $(T\Psi v, \Psi v)$ instead of $(T^{\frac{1}{2}}\Psi v, T^{\frac{1}{2}}\Psi v)$. But they only differ for

$$\left| \int_M e^{-\phi} \left([c_{ij}, T^{\frac{1}{2}}](T^{\frac{1}{2}}\Psi v, \Psi v) \right) dV \right| \lesssim ||\Psi v||_0^2,$$

which is negligible.

□

We go back to the family of weights of Theorem 1.1 and Proposition 2.3. We apply (3.2) (resp. (3.3)) for $\phi = \phi^k + t|z'|^2$ (resp. $\phi = \phi^k - t|z'|^2$). First, we note that they are absolutely uniformly bounded with respect to k ; they can be made bounded in t by taking $U = \{z : |z'| < \frac{1}{t}\}$. (In particular, by boundedness, they can be removed from the norms.) Possibly by raising to exponential, boundedness implies “selfboundedness of the gradient” when ϕ is plurisubharmonic. In our case, in which to be positive is not (ϕ_{ij}^k) itself but $2^k(c_{ij}) + (\phi_{ij}^k)$, we have, for $|z'|$ small

$$\begin{aligned}
|\partial_b \phi|^2 & = |\partial_b(\phi^k + t|z'|^2)|^2 \\
& \lesssim |\partial_b \phi^k|^2 + t^2 |z'|^2 \\
& \lesssim 2^k(c_{ij}) + (\phi_{ij}^k) + t. \quad (3.16)
\end{aligned}$$

So $\|\partial_b \phi \lrcorner \Psi u^\pm\|^2$ can be removed from the right side of both (3.2) and (3.3). Also, the term $-\frac{1}{2}(c_{ij})T(\phi)(v, \bar{v})$ is controlled by $(c_{ij})(T^{\frac{1}{2}}v, \overline{T^{\frac{1}{2}}v})$ by Sobolev interpolation. We then combine Proposition 2.3 with Theorem 3.1 formula (3.2) for the weight $\phi^k + t|z'|^2$ (resp. formula (3.10) for the weight $(\phi^k - t|z'|^2)$), and notice that $T^{\frac{1}{2}} \sim \Lambda^{\frac{1}{2}}$ on $\text{supp } \Psi^+$ (resp. $(T^-)^{\frac{1}{2}} \sim \Lambda^{\frac{1}{2}}$ on $\text{supp } \Psi^-$). Also, on the right of (3.2) and (3.3), one reduces $\|\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}v\|_\phi^2$ to $\|v\|_\phi^2$ by induction and estimates all terms $Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^\phi$ and $Q_{\text{Op}^{\text{ord}(\Psi)-j}}^\phi$ $j \geq 1$ by a common $Q_{\Psi'}^\phi$.

Proof of Theorem 1.3. We have to use (3.2) with the above choice of the weight ϕ and take summation over k ; this yields (1.3) for $v = u^+$. The twin estimate for $v = u^-$ follows from (3.3) by similar procedure. Finally, (1.4) comes as the combination of (1.3) for $v = u^+$, the twin for $v = u^-$ and the elliptic estimate for $v = u^0$.

□

4. A CRITERION OF HYPOELLIPTICITY OF THE KOHN LAPLACIAN

Let M be a pseudoconvex, hypersurface type manifold of \mathbb{C}^n , $\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ the Kohn Laplacian of M , and $G := \square_b^{-1}$ the Green operator.

Proof of Theorem 1.4 Our program is to prove that for any cut-off $\eta_o \in C_c^\infty(U)$ with $\eta_o \equiv 1$ in a neighborhood of z_o , for suitable $\eta \succ \eta_o$, that is $\eta|_{\text{supp } \eta_o} \equiv 1$, for any s and suitable U , we have

$$\|\eta_o u\|_s \lesssim \|\eta \bar{\partial}_b u\|_s + \|\eta \bar{\partial}_b^* u\|_s + \|u\|_0 \quad \text{for any } u \in \mathcal{H}^\perp \cap C^\infty(M \cap U)$$

$$\text{in any degree } k \in [0, n-1]. \quad (4.1)$$

If we are able to prove (4.1), we have immediately the exact local H^s -regularity of $\bar{\partial}_b^* G$ and $\bar{\partial} G$ over $\ker \bar{\partial}$ and $\ker \bar{\partial}^*$ respectively. From this, we get the (non-exact) regularity of the Szegö $S = \text{Id} - \bar{\partial}_b^* G \bar{\partial}_b$ and anti-Szegö $S^* = \text{Id} - \bar{\partial}_b G \bar{\partial}_b^*$ projection respectively. (At this stage we need to apply the method of the elliptic regularization to pass from C^∞ - to H^s -forms.) From this the (non-exact) regularity of G itself follows (cf. e.g. the proof of Theorem 2.1 of [1]). Along with $\eta_o \prec \eta$, we consider an additional cut-off σ with $\eta_o \prec \sigma \prec \eta$ and denote by R^s the pseudodifferential operator with symbol $(1 + |\xi|^2)^{\frac{s\sigma(a)}{2}}$. According to Proposition 2.1 of [1], there is no restriction on the degree of u ; thus u can be either a form or a function. By Section 3 above, we can prove (4.1) separately on each term of the microlocal decomposition of $u = u^+ + u^- + u^0$; since u^0 has elliptic estimate and u^- can be reduced to u^+ by star-Hodge correspondence, we prove the result only for

$v = u^+$. We start from

$$\begin{aligned}
\|\Lambda^s \eta_o v\| &\lesssim \|R^s \eta_o v\| + \|v\| \\
&= \|R^s \eta_o \eta^2 v\| + \|v\| \\
&\leq \|R^s \eta^2 v\| + \|[R^s, \eta_o] \eta^2 v\| + \|v\| \\
&\lesssim \|R^s \eta^2 v\| + \|v\| \\
&\lesssim \|\eta R^s \eta v\| + \|[R^s, \eta] \eta v\| + \|v\| \\
&\lesssim \|\eta R^s \eta v\| + \|v\|,
\end{aligned} \tag{4.2}$$

(cf. [15] Section 7). Next, we apply Theorem 3.1 for $\Psi = \eta R^s \eta$. What we have to describe are the error terms in the right of (3.2), (3.3), that is, $[\partial_b, \eta R^s \eta]$ and $[\partial_b, [\bar{\partial}_b, \eta R^s \eta]]$. Since the argument is similar for the two, we only treat the first. We have by Jacobi identity

$$\begin{aligned}
[\partial_b, \eta R^s \eta] &= [\partial_b, \eta] R^s \eta + \eta [\partial_b, R^s] \eta + \eta R^s [\partial_b, \eta] \\
&= [\partial_b, R^s] + \text{Op}^{-\infty}.
\end{aligned} \tag{4.3}$$

In fact, since $\text{supp } \partial_b \eta \cap \text{supp } \sigma = \emptyset$, then the first and last terms in the right of the first line of (4.3) are operators of order $-\infty$ and can therefore be disregarded. As for the central term, we have

$$[\partial_b, R^s] = \partial_b(\sigma) \log(\Lambda) R^s. \tag{4.4}$$

Now, our hypothesis is that

$$\begin{aligned}
\|\log(\Lambda) \partial_b \sigma \lrcorner \eta R^s \eta v\|^2 &\leq \epsilon \left(\int_M (c_{ij}) (\Lambda^{\frac{1}{2}} \eta R^s \eta v, \overline{\Lambda^{\frac{1}{2}} \eta R^s \eta v}) dV \right. \\
&\quad \left. + \sum_{k=1}^{+\infty} \int_M \left((\phi_{ij}^k) (\eta R^s \eta \Gamma_k v, \overline{\eta R^s \eta \Gamma_k v}) \right) dV \right) + c_\epsilon \|\eta R^s \eta v\|^2.
\end{aligned} \tag{4.5}$$

Altogether, we get

$$\begin{aligned}
t||\Lambda^s \eta_o v||^2 &\stackrel{\sim}{\leq} t||\eta R^s \eta v||_0^2 + ||v||_0^2 \\
(4.2) \quad &\leq \left(\int_M (c_{ij})(\Lambda^{\frac{1}{2}} \Psi v, \overline{\Lambda^{\frac{1}{2}} \Psi v}) dV + \sum_{k=1}^{+\infty} \int_M (\phi_{ij}^k)(\Gamma_k \Psi v, \overline{\Gamma_k \Psi v}) dV \right) + t||\eta R^s \eta v||_0^2 + c_\epsilon ||v||_0^2 \\
&\stackrel{\sim}{\leq} Q_{\eta R^s \eta}^b(v, \bar{v}) + ||[\partial_b, \eta R^s \eta] \perp v||_0^2 + \left| \int_M [\partial_b, [\bar{\partial}_b, \eta R^s \eta]](v, \bar{v}) dV \right| \\
&\text{by the second of (1.3)} \\
&+ \left| \sum_h \int (c_{ij}^h)([\partial_{\omega_h}, \Psi](v), \overline{\Psi v}) dV \right| + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^b(v, \bar{v}) + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} v||_0^2 \\
&\stackrel{\sim}{\leq} Q_{\eta R^s \eta}^b(v, \bar{v}) + ||\partial_b(\sigma) \log(\Lambda) \eta R^s \eta v||_0^2 + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^b(v, \bar{v}) + ||\eta' v||_{s-\epsilon}^2 \\
(4.4) \text{ and (1.6) (c)} \quad &\stackrel{\sim}{\leq} Q_{\eta R^s \eta}^b(v, \bar{v}) + \epsilon \left(\int_M (c_{ij})(\Lambda^{\frac{1}{2}} \eta R^s \eta v, \overline{\Lambda^{\frac{1}{2}} \eta R^s \eta v}) dV + \sum_k \int \left((\phi_{ij}^k)(\eta R^s \eta \Gamma_k v, \times \right. \right. \\
(4.5) \quad &\times \overline{\eta R^s \eta \Gamma_k v}) \left. \right) dV \Big) + c_\epsilon ||\eta R^s \eta v||_0^2 + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^b(v, \bar{v}) + ||\eta' v||_{s-\epsilon}^2 \\
&\stackrel{\sim}{\leq} Q_{\eta R^s \eta}^b(v, \bar{v}) + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^b(v, \bar{v}) + ||\eta R^s \eta v||_0^2 + ||\eta' v||_{s-\epsilon}^2 \\
&\text{absorbtion in the second line} \\
&\stackrel{\sim}{\leq} Q_{\eta R^s \eta}^b(v, \bar{v}) + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}^b(v, \bar{v}) + ||\eta' v||_{s-\epsilon}^2. \\
&\text{absorbtion by means of } t
\end{aligned} \tag{4.6}$$

Now, the $s - \epsilon$ norm is reduced to 0 norm by induction over j with $j\epsilon > s$, and $Q_{\eta R^s \eta}$ and the various $Q_{\text{Op}^{s-\epsilon j-1}}$ are estimated by a common $Q_{\eta' \Lambda^s}$. In conclusion, we have got (4.1) with the notational difference of η' instead of η .

□

Proof of Theorem 1.5. We choose our cut-off starting from a cut-off χ in one real variable and setting $\eta = \Pi_j \chi(|z_{Ij}|) \chi(|y_n|)$. We have

- (a) $\text{supp } \partial_{z_{Ij}} \chi(|z_{Ij}|)$ is contained in $z_{Ij} \neq 0$ in particular, outside the “critical” curve Γ where superlogarithmic estimates hold by Theorem 1.1 and Theorem 1.2; thus $\partial_b(\Pi_j \chi(|z_{Ij}|))$ are superlogarithmic multipliers.
- (b) $\partial_b \chi(|y_n|) \sim \cdot (h_{z_{Ij}}^{Ij})_j$ and hence it is by hypothesis a superlogarithmic multiplier.

Altogether, $\partial_b \eta \perp$ are superlogarithmic multipliers. Remember that we are assuming that (c_{ij}^h) are subelliptic multipliers. Finally, $\text{supp } [\partial_b, [\bar{\partial}_b, \chi(z_{Ij})]]$ is contained in $z_{Ij} \neq 0$ and

$[\partial_b, [\bar{\partial}_b, \chi(y_n)]] \sim h_{z_{IJ}, \bar{z}_{IJ}}^{I^j}$ are subelliptic multipliers; in conclusion, $[\partial_b, [\bar{\partial}_b, \eta]]$ are superlogarithmic multipliers. We can then apply Theorem 1.4 and this completes the proof of Theorem 1.5

□

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